

Memory effects in quantum information transmission across a Hamiltonian dephasing channel

A. D'Arrigo^{1,a}, G. Benenti^{2,3,b}, and G. Falci^{1,c}

¹ MATIS CNR-INFM, Catania & Dipartimento di Metodologie Fisiche e Chimiche per l'Ingegneria, Università degli Studi di Catania, Viale Andrea Doria 6, 95125 Catania, Italy

² CNISM, CNR-INFM & Center for Nonlinear and Complex Systems, Università degli Studi dell'Insubria, Via Valleggio 11, 22100 Como, Italy

³ Istituto Nazionale di Fisica Nucleare, Sezione di Milano, via Celoria 16, 20133 Milano, Italy

Abstract. We study a dephasing channel with memory, modelled by a multimode environment of oscillators. Focusing on the case of two channel uses, we show that memory effects can enhance the amount of coherent quantum information transmitted down the channel. We also show the Kraus representation for two channel uses. Finally, we propose a coding-decoding scheme that takes advantage of memory to improve the fidelity of transmission.

1 Introduction

Quantum communication channels [1,2] use quantum systems to transfer classical or quantum information. In the first case, classical bits are encoded by means of quantum states. In the latter, one can transfer an unknown quantum state between different units of a quantum computer, or distribute entanglement between two or more communicating parties. The fundamental quantities characterizing a quantum channel are the *classical* and the *quantum channel capacities*, that are defined as the maximum number of bits/qubits that can be reliably transmitted per channel use [3].

Quantum channels are the natural theoretical framework to investigate both quantum communication and computation in a noisy environment. In the first case, information is transmitted in space, in the latter in time. In both cases, noise can have relevant low frequency components, which traduce themselves in memory effects. That is, consecutive uses of a channel can be correlated. Memory effects may be important, for instance, in quantum communication protocols realized by means of photons travelling across fibers with birefringence fluctuating with characteristic time scales longer than the separation between successive light pulses [4]. Moreover, solid state implementations of quantum hardware show a characteristic low-frequency noise [5].

Previous investigations have shown that memory can enhance classical information transmission down a quantum memory channel [6]. More recently, we have considered a channel subject to dephasing noise described by a Markov chain with memory effects, showing that the quantum capacity increases with respect to the memoryless case [7]. Furthermore, based on theoretical arguments and numerical simulations, we have conjectured that the enhancement of the quantum capacity also takes place for a dephasing quantum environment modelled by a bosonic bath [7]. In this paper, we discuss memory effect in this latter model and address the case of two channel uses. This is interesting because, while channel capacity is defined in the

^a e-mail: antonio.darrigo@dmfci.unict.it

^b e-mail: giuliano.benenti@uninsubria.it

^c e-mail: gfalci@dmfci.unict.it

asymptotic limit of an infinite number of uses, real coding-decoding schemes [1,2] necessarily work on a finite number of uses. The case of two channel uses nicely show that memory effects may enhance the coherent quantum information transmitted down the channel. Furthermore, we propose a simple coding-decoding scheme that takes advantage of memory to improve the fidelity of transmission.

Dephasing channels are characterized by the property that when N qubits are sent through the channel, the states of a preferential orthonormal basis $\{|j\rangle \equiv |j_1, \dots, j_N\rangle, j_1, \dots, j_N = 0, 1\}$ are transmitted without errors. Therefore, dephasing channels are noiseless from the viewpoint of the transmission of classical information, since the states of the preferential basis can be used for encoding classical information. Of course superpositions of basis states may decohere, thus corrupting the transmission of quantum information. We point out that dephasing channels are relevant for systems in which relaxation is much slower than dephasing [8,9].

The paper is organized as follows. In Sec. 2 we review basic quantities useful to describe quantum information transmission across a noisy channel. In Sec. 3 we introduce the Hamiltonian dephasing channel investigated in the paper, while in Sec. 4 we recall some known results for a single channel use, relevant to describe the memoryless limit. Finally, in Sec. 5 we study the case of two channel uses. In particular, we report the channel transformation for a generic input (Sec. 5.1) and the corresponding Kraus representation (Sec. 5.2). We also discuss memory effects for a class of input states (Sec. 5.3) and finally propose a simple and efficient coding/decoding scheme that takes advantage of memory (Sec. 5.4). Our conclusions are in Sec. 6.

2 Quantum information transmission: basic definitions

In this section, we briefly review basic quantities and concepts that are useful to describe the channel ability to transmit quantum information. For this purpose, we consider as quantum information carrier a quantum system \mathcal{Q} and describe the channel action on \mathcal{Q} by means of a superoperator [1,2] \mathcal{E} , that is, a completely positive, trace preserving linear map that transforms the (generally mixed) input state $\rho^{\mathcal{Q}}$ into the output state $\rho^{\mathcal{Q}'}$. In a prototype communication protocol, \mathcal{Q} may be an N -qubit system representing N successive uses of the channel.

2.1 Quantum information

Quantum information is the information related to a quantum source [10,11], that is, a source Σ of identical quantum systems \mathcal{Q} , which are prepared in an unknown quantum state chosen from the ensemble $\{\rho_k\}$, according to a given stationary probability distribution $\{p_k\}$. The amount of information generated by the source is measured by the von Neumann entropy $S(\Sigma) \equiv S(\rho) = -\text{Tr}[\rho \log_2 \rho]$, where $\rho = \sum_k p_k \rho_k$ [10,11]. Indeed, $S(\Sigma)$ provides the number of quantum two level systems necessary to efficiently encode the source Σ according to the noiseless quantum coding theorem. Quantum information reduces to classical information only if the states $\{\rho_k\}$ are pure and mutually orthogonal.

2.2 Reliable transmission and entanglement fidelity

A proper way to measure reliability of quantum information transmission is the entanglement fidelity [12,13]. To define this quantity we look at the system \mathcal{Q} as a part of a larger quantum system \mathcal{RQ} , initially in a pure entangled state $|\psi^{\mathcal{RQ}}\rangle$. The density operator of the system \mathcal{Q} is then obtained from that of \mathcal{RQ} by a partial trace over \mathcal{R} : $\rho^{\mathcal{Q}} = \text{Tr}_{\mathcal{R}}[|\psi^{\mathcal{RQ}}\rangle\langle\psi^{\mathcal{RQ}}|]$. The system \mathcal{Q} is sent through the channel and undergoes the transformation \mathcal{E} , while \mathcal{R} is ideally isolated from the environment (see Fig. (1) left). The final state of the composite system is:

$$\rho^{\mathcal{RQ}'} = (\mathcal{I}^{\mathcal{R}} \otimes \mathcal{E}^{\mathcal{Q}})(|\psi^{\mathcal{RQ}}\rangle\langle\psi^{\mathcal{RQ}}|), \quad (1)$$

where \mathcal{I} is the identity superoperator. Let us consider the following question: *how faithfully*

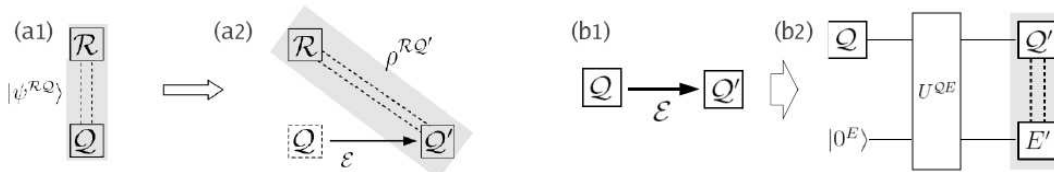


Fig. 1. Left: The system \mathcal{Q} is considered as entangled with a reference system \mathcal{R} , so that the initial state $|\psi^{\mathcal{RQ}}\rangle$ of the overall system \mathcal{RQ} is pure; then (a2) the system \mathcal{Q} is sent through the channel which affects both the system \mathcal{Q} and the entanglement between \mathcal{Q} and \mathcal{R} . Right: Sketch of the equivalence between the map \mathcal{E} (b1) and the unitary representation in (b2), in which the effect of the map \mathcal{E} on the system is reproduced by a unitary evolution $U^{\mathcal{QE}}$ of the system itself plus a fictitious environment E , which is initially in a pure state.

does the channel preserve the entanglement between the two systems \mathcal{Q} and \mathcal{R} ? The answer is the *entanglement fidelity* F_e [12], defined as the fidelity F between the initial pure state $|\psi^{\mathcal{RQ}}\rangle$ and the final (generally mixed) state $\rho^{\mathcal{RQ}'}$:

$$F_e = F_e(\rho^{\mathcal{Q}}, \mathcal{E}) = F(|\psi^{\mathcal{RQ}}\rangle, \rho^{\mathcal{RQ}'}) = \langle \psi^{\mathcal{RQ}} | (\mathcal{I}^{\mathcal{R}} \otimes \mathcal{E}^{\mathcal{Q}}) (|\psi^{\mathcal{RQ}}\rangle \langle \psi^{\mathcal{RQ}}|) | \psi^{\mathcal{RQ}} \rangle. \quad (2)$$

It can be shown that entanglement fidelity only depends on the initial state $\rho^{\mathcal{Q}}$ of the system and on the channel action \mathcal{E} , not on the particular purification $|\psi^{\mathcal{RQ}}\rangle$ chosen [12]. Unlike the usual input-output fidelity [1,2], entanglement fidelity looks at the same physical process, the transmission of \mathcal{Q} across a channel, from a different point of view: as a local transformation on a part (\mathcal{Q}) of an entangled system (\mathcal{RQ}). This transformation is undesired because it can reduce the amount of entanglement of the overall system.

2.3 Channel noise and entropy exchange

The entropy exchange [12] is the entropy that the enlarged system \mathcal{RQ} acquires when \mathcal{Q} undergoes the transformation \mathcal{E} :

$$S_e = S_e(\rho^{\mathcal{Q}}, \mathcal{E}) = S(\rho^{\mathcal{RQ}'}), \quad (3)$$

where $\rho^{\mathcal{RQ}'}$ is given by (1). It can be shown that the entropy exchange is an intrinsic function of the system input $\rho^{\mathcal{Q}}$ and of the channel \mathcal{E} and does not depend on the particular purification [12]. Since \mathcal{E} is a completely positive trace preserving linear map, it can be represented [1,2] by a unitary evolution $U^{\mathcal{QE}}$ on a larger system, given by the system \mathcal{Q} itself plus an ancilla E , that is, a (generally fictitious) environment initially in a pure state (see Fig. (1) right). It is straightforward to show that [12,14]:

$$S(\rho^{\mathcal{RQ}'}) = S(\rho^{E'}) = S_e, \quad (4)$$

where $\rho^{E'}$ is the final state of the environment. S_e measures the entropy increase of the environment E or, equivalently, the entanglement between \mathcal{RQ} and E after the evolution U . The entropy exchange can be thought of as the quantum analogue of the classical *conditional entropy* [15,16], that measures the average uncertainty about the channel output for a known input, namely the noise added to the output by the channel.

2.4 Quantum capacity and coherent information

Classical channel capacity is given by the maximum (over the input probability distribution) of the input-output *mutual information* [15,16]. The quantity analogous to mutual information for quantum information is the *coherent information* I_c [14,17], defined as

$$I_c = I_c(\rho^{\mathcal{Q}}, \mathcal{E}) = S(\mathcal{E}(\rho^{\mathcal{Q}})) - S_e(\rho^{\mathcal{Q}}, \mathcal{E}) = S(\rho^{\mathcal{Q}'}) - S(\rho^{\mathcal{RQ}'}). \quad (5)$$

For memoryless channels and for the so-called forgetful channels [18], for which memory effects decay exponentially with time, the *quantum channel capacity* Q is the maximum of I_c/N over all possible input states, in the limit of number of channel uses $N \rightarrow \infty$ [19,20,21,22]. Note that the coherent information is maximal when \mathcal{Q} and \mathcal{R} are maximally entangled and the channel is noiseless. Indeed, in this case $S(\rho^{\mathcal{Q}'}) = S(\rho^{\mathcal{Q}})$ is maximal because the input state $\rho^{\mathcal{Q}}$ is maximally mixed, and $S(\rho^{\mathcal{R}\mathcal{Q}'}) = 0$, since the state $|\psi^{\mathcal{R}\mathcal{Q}}\rangle$ remains pure after the transmission. Smaller values of the coherent information are obtained when the state $|\psi^{\mathcal{R}\mathcal{Q}}\rangle$ is not maximally entangled or noise affects the channel. This example illustrates the fact that coherent information measures the possibility to convey entanglement through a communication channel.

3 Channel Hamiltonian model

We suppose that information is carried by qubits that transit across a communication channel, modeled as a purely dephasing environment. The Hamiltonian describing the transmission of N qubits through the channel reads [7]

$$H(t) = H_E - \frac{1}{2}X_E F(t), \quad F(t) = \lambda \sum_{k=1}^N \sigma_z^{(k)} f_k(t), \quad (6)$$

where H_E is the environment Hamiltonian and X_E the environment coupling operator. The k -th qubit is coupled to the environment via its Pauli operator $\sigma_z^{(k)}$, the coupling strength being λ . The functions $f_k(t)$ switch on and off the coupling: $f_k(t) = 1$ when the k -th qubit is inside the channel, $f_k(t) = 0$ otherwise. We call τ_p the time each carrier takes to cross the channel and τ the time interval that separates two consecutive qubits entering the channel. Note that Hamiltonian (6) is expressed in the interaction picture with respect to the qubits.

We call ω_0 and $\rho^{\mathcal{Q}}$ the density operators which represent the initial states of the environment and of the N qubits, respectively. Assuming that initially the system and the environment are not entangled, we can write the state of the system at time t as follows:

$$\rho^{\mathcal{Q}}(t) = \text{Tr}_E\{U(t)(\rho^{\mathcal{Q}} \otimes \omega_0)U^\dagger(t)\}, \quad (7)$$

where

$$U(t) = T e^{-\frac{i}{\hbar} \int_0^t ds H(s)}. \quad (8)$$

In particular, we are interested in the final state $\rho^{\mathcal{Q}'}$ after all N qubits crossed the channels. To treat this problem we choose the factorized basis states $\{|j\rangle_{\alpha_E}\}$, where $\{|j\rangle = |j_1, \dots, j_N\rangle\}$ are the eigenvectors of $\prod_k \sigma_z^{(k)}$, and $\{|\alpha_E\rangle\}$ is an orthonormal basis for the environment. The dynamics preserves the basis states $|j\rangle$ and therefore the evolution operator (8) is diagonal in the system indices:

$$\langle j\rangle_{\alpha_E} U(t) |l\rangle_{\alpha'_E} = \delta_{jl} \langle \alpha_E | U(t) | j \rangle | \alpha'_E \rangle, \quad (9)$$

where $U(t)|j\rangle = \langle j|U(t)|j\rangle$ expresses the conditional evolution operator of the environment alone. Therefore,

$$(\rho^{\mathcal{Q}'})_{jl} = (\rho^{\mathcal{Q}})_{jl} \sum_{\alpha} \langle \alpha_E | U(t) | j \rangle w_0 U^\dagger(t) | l \rangle | \alpha_E \rangle. \quad (10)$$

In this basis representation the populations are preserved and the environment only changes the off-diagonal elements of $\rho^{\mathcal{Q}}$.

Now we model the environment with an infinite set of oscillators:

$$H_E = \sum_{\alpha} \omega_{\alpha} b_{\alpha}^{\dagger} b_{\alpha} + H_C, \quad H_C = \sum_{\alpha} \frac{\lambda^2}{4\omega_{\alpha}} \sum_{k=1}^N \sigma_z^{(k)}, \quad X_E = \sum_{\alpha} (b_{\alpha}^{\dagger} + b_{\alpha}), \quad (11)$$

where H_C is a counterterm [23]. If the environment is initially in thermal equilibrium, $w_0 = e^{-\beta \mathcal{H}_E}$, we obtain [7]

$$\sum_{\alpha} \langle \alpha_E | U(t|j) w_0 U^\dagger(t|l) | \alpha_E \rangle = \exp \left[-\lambda^2 \int_0^\infty \frac{dw}{\pi} S(w) \frac{1 - \cos(w\tau_p)}{w^2} \left| \sum_{k=1}^N (j_k - l_k) e^{i\omega(k-1)\tau} \right|^2 \right], \quad (12)$$

where $S(\omega)$ is the power spectrum of the coupling operator X_E , that is the Fourier transform of the bath symmetrized autocorrelation function: $C(t) = 1/2 \langle X_E(t)X_E(0) + X_E(0)X_E(t) \rangle$.

4 Single channel use

In this section, we briefly discuss the case in which a single qubit is sent down the Hamiltonian channel (6),(11). The qubit states before and after the channel transmission read as follows:

$$\rho^{\mathcal{Q}} = \frac{1}{2} \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} \Rightarrow \rho^{\mathcal{Q}'} = \mathcal{E}_1(\rho^{\mathcal{Q}}) = \frac{1}{2} \begin{pmatrix} \rho_{00} & g\rho_{01} \\ g\rho_{10} & \rho_{11} \end{pmatrix}, \quad (13)$$

\mathcal{E}_1 is the map for single channel use. The dephasing factor $g \in [0, 1]$ is deduced from (12) for $N = 1$:

$$g = \exp \left\{ -\lambda^2 \int_0^\infty \frac{dw}{\pi} S(w) \frac{1 - \cos(w\tau_p)}{w^2} \right\}. \quad (14)$$

It is possible to give a simple representation of the channel action in terms of Kraus operators [1,2]:

$$\mathcal{E}_1(\rho^{\mathcal{Q}}) = \sum_{m \in \{0,z\}} p_m B_m \rho^{\mathcal{Q}} B_m^\dagger, \quad B_m = \sigma_m \quad (15)$$

where $\sigma_0 = \mathbf{1}$, $p_0 = (1+g)/2$, and $p_z = (1-g)/2$.

Using the Kraus form (15), we can compute F_e for a generic input state [12]:

$$F_e(\rho^{\mathcal{Q}}, \mathcal{E}_1) = \sum_m |\text{Tr}_{\mathcal{Q}}[\rho^{\mathcal{Q}} A_m^{\mathcal{Q}}]|^2 = \frac{1+g}{2} + \frac{1-g}{2} z^2, \quad (16)$$

where the Bloch coordinate $z = \rho_{00} - \rho_{11}$ is the expectation value of σ_z , and $A_m = \sqrt{p_m} B_m$. Note that F_e is independent of the coherences ρ_{01} . In particular, we have

$$F_e^{(z=0)} = \frac{1+g}{2}. \quad (17)$$

This case is relevant as it takes place when Alice possesses a maximally entangled (Bell) pair \mathcal{RQ} and sends a member of the pair to Bob. The qubit (\mathcal{Q}) sent to Bob is in the maximally mixed state $\rho^{\mathcal{Q}} = \frac{1}{2} \mathbf{1}$, therefore $z = 0$ and Alice and Bob eventually share a pair whose fidelity is given by $F_e^{(z=0)}$. This implies that a Bell measurement able to distinguish the ideally shared Bell state from the other states of the Bell basis fails with *error probability* $P_e = 1 - F_e$.

The entropy exchange $S_e(\rho^{\mathcal{Q}}, \mathcal{E}_1)$ can be computed as the von Neumann entropy of the matrix $W_{mn} = \text{Tr}_{\mathcal{Q}}[A_m^{\mathcal{Q}} \rho^{\mathcal{Q}} A_n^{\dagger \mathcal{Q}}]$ [12]. The eigenvalues of W are $\lambda_{1,2}^W = \frac{1}{2}(1 \pm \sqrt{g^2 + (1-g^2)z^2})$. Then the entropy exchange is given by

$$S_e = S(W) = - \sum_{m=1}^2 \lambda_m^W \cdot \log_2 \lambda_m^W. \quad (18)$$

In particular,

$$S_e^{(z=0)} = H\left(\frac{1+g}{2}\right) = H(F_e^{(z=0)}), \quad (19)$$

where $H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ is the Shannon binary entropy. It is worth noticing that the entropy exchange takes its maximum for the completely dephasing channel ($g = 0$), for which the entanglement fidelity F_e is minimum.

The coherent information is given by $I_c(\rho^{\mathcal{Q}}, \mathcal{E}_1) = H(\lambda_1^{\text{out}}) - H(\lambda_1^W)$, where $\lambda_{1,2}^{\text{out}} = \frac{1}{2}(1 \pm \sqrt{z^2 + g^2 \gamma^2})$ are the eigenvalues of $\rho^{\mathcal{Q}'}$. Here $\gamma^2 = x^2 + y^2$, the Bloch coordinates $x = 2\text{Re}(\rho_{01})$ and $y = -2\text{Im}(\rho_{01})$ being the expectation values of σ_x and σ_y . It is easy to show that the coherent information is maximized by the input state $\rho^{\mathcal{Q}} = \frac{1}{2}\mathbf{1}$, that is, for $x = y = z = 0$. In this case,

$$I_c^{(z=\gamma=0)} = 1 - S_e^{(z=0)} = 1 - H\left(\frac{1+g}{2}\right). \quad (20)$$

It can be proved that (20) is the quantum capacity for a memoryless dephasing channel [24,25,26].

5 Memory dephasing channel: two channel uses

In this section, we consider two channel uses. Provided that the time τ between two channel uses is smaller than the time scale τ_c associated with the decay of environmental correlation functions, the action of the environment on the second qubit is related to the action on the first qubit. Therefore, $\mathcal{E}_2 \neq \mathcal{E}_1 \otimes \mathcal{E}_1$, where the superoperator \mathcal{E}_2 describes the transformation operated by the channel on the overall two-qubit system. We show that channel memory can enhance the coherent information. Moreover, one can exploit memory effects to design suitable coding-decoding schemes that improve the faithfulness of quantum information transmission.

5.1 The system transformation

We consider the transmission of two qubits, \mathcal{Q}_1 and \mathcal{Q}_2 , initially in the state $\rho^{\mathcal{Q}_1 \mathcal{Q}_2}$, with matrix elements ρ_{mn} , $m, n = 0, \dots, 3$. The final state of the system is

$$\rho^{\mathcal{Q}'_1 \mathcal{Q}'_2} = \mathcal{E}_2(\rho^{\mathcal{Q}_1 \mathcal{Q}_2}) = \begin{pmatrix} \rho_{00} & g \cdot \rho_{01} & g \cdot \rho_{02} & h^+ \cdot \rho_{03} \\ g \cdot \rho_{10} & \rho_{11} & h^- \cdot \rho_{12} & g \cdot \rho_{13} \\ g \cdot \rho_{20} & h^- \cdot \rho_{21} & \rho_{22} & g \cdot \rho_{23} \\ h^+ \cdot \rho_{30} & g \cdot \rho_{31} & g \cdot \rho_{32} & \rho_{33} \end{pmatrix}, \quad (21)$$

where the factors g and h^\pm describe the channel effects and the noiseless limit is recovered for $g = h^\pm = 1$. The last two terms are defined as

$$h^\pm = \exp \left\{ -2\lambda^2 \int_0^\infty \frac{dw}{\pi} S(w) \frac{1 - \cos(w\tau_p)}{w^2} (1 \pm \cos w\tau) \right\} \quad (22)$$

and are derived from (12) for $j_1 = j_2 = j$, $l_1 = l_2 \neq j$ (h^+) and for $j_1 = l_2 = j$, $j_2 = l_1 \neq j$ (h^-).

In the absence of any memory effects, that is, when the power spectrum $S(\omega)$ is white and there is no superposition between the time windows when the first or the second qubits are inside the channel ($\tau \geq \tau_p$), we have $h^\pm = g^2$; therefore, $\mathcal{E}_2 = \mathcal{E}_1 \otimes \mathcal{E}_1$. In the opposite limiting case of perfect memory, that is, when $\tau_c \gg \tau, \tau_p$, or alternatively the two time windows of the qubit-environment interaction are completely superimposed ($\tau = 0$), we have $h^+ = g^4$ and $h^- = 1$. In this limit the subspace spanned by the basis $\{|00\rangle, |11\rangle\}$ undergoes a stronger decoherence (with respect to the memoryless case), while the subspace spanned by $\{|01\rangle, |10\rangle\}$ is *decoherence free*.

It is convenient to measure the memory between two channel uses by introducing the memory coefficient γ defined as follows:

$$\gamma = \int_0^\infty \frac{dw}{\pi} S(w) \frac{1 - \cos(w\tau_p)}{w^2} \cos w\tau \bigg/ \int_0^\infty \frac{dw'}{\pi} S(w') \frac{1 - \cos(w'\tau_p)}{w'^2}. \quad (23)$$

For a given power spectrum $S(\omega)$ and crossing time τ_p , γ only depends on the time interval τ , and ranges in the interval $[0, 1]$. In particular, $\gamma = 0$ for a memoryless channel (as it can be checked by letting $\tau \rightarrow \infty$ in the (23)), while $\gamma = 1$ for perfect memory ($\tau = 0$ in (23)). We can express the dephasing factors h^\pm by means of the corresponding memoryless value g^2 and the memory factor γ :

$$h^\pm = g^{2(1 \pm \gamma)}. \quad (24)$$

5.2 Kraus representation

In presence of memory ($\gamma > 0$), the Kraus representation in our Hamiltonian model cannot be trivially derived from (15) by simply concatenating the Kraus operators for single channel use with a suitable probability distribution:

$$\mathcal{E}_2(\rho^{\mathcal{Q}_1 \mathcal{Q}_2}) \neq \sum_{m_1 m_2 \in \{0, z\}} p(m_2, m_1) B_{m_1} \otimes B_{m_2} \rho^{\mathcal{Q}_1 \mathcal{Q}_2} B_{m_1}^\dagger \otimes B_{m_2}^\dagger, \quad (25)$$

where the B_m operators are the same as in (15). It is worth noting that in (25) memory could be taken into account by means of the joint probability $p(m_2, m_1) = p(m_2|m_1)p(m_1)$, where the conditional probability $p(m_2|m_1)$ can introduce a correlation between the occurrence of the second qubit operator B_{m_2} and the action of first qubit operator B_{m_1} . For simplicity we rename all possible combinations of the B_m operators in (25): $K_0 \equiv B_0 \otimes B_0$, $K_1 \equiv B_0 \otimes B_z$, $K_2 \equiv B_z \otimes B_0$ and $K_3 \equiv B_z \otimes B_z$. It turns out that the Kraus representation for the map eq. (21) requires two other operators $K_4 \equiv \frac{1}{2}(K_1 + K_2)$ and $K_5 \equiv \frac{1}{2}(K_0 - K_3)$:

$$\mathcal{E}_2(\rho^{\mathcal{Q}_1 \mathcal{Q}_2}) = \sum_{m=0}^5 p_{K_m} K_m \rho^{\mathcal{Q}_1 \mathcal{Q}_2} K_m^\dagger \quad (26)$$

where $p_{K_0} = \frac{1}{4}(1 + 2g + h^+)$, $p_{K_1} = p_{K_2} = \frac{1}{4}(1 - h^-)$, $p_{K_3} = \frac{1}{4}(1 - 2g + h^+)$ and $p_{K_4} = p_{K_5} = \frac{1}{4}(h^- - h^+)$. For $\gamma = 0$, (26) is exactly equivalent to concatenating twice (15), namely $\mathcal{E}_2 = \mathcal{E}_1 \otimes \mathcal{E}_1$. The behaviour of the (25) and (26) are very different. In fact for perfect memory the first Kraus representation, in which we have to set $p(m_2|m_1) = \delta_{m_2, m_1}$, generates two decoherence free subspaces: $\{|00\rangle, |11\rangle\}$ as well as $\{|01\rangle, |10\rangle\}$.

5.3 Entanglement fidelity, entropy exchange and coherent information

The purpose of this section is to show that memory effects can improve the channel performance. We consider diagonal input states of the type

$$\rho^{\mathcal{Q}_1 \mathcal{Q}_2} = \frac{1}{4} [p(|01\rangle\langle 01| + |10\rangle\langle 10|) + q(|00\rangle\langle 00| + |11\rangle\langle 11|)]. \quad (27)$$

This density operator describes a mixture in which we have, with probability p , the unpolarized state of the subspace spanned by $\{|01\rangle, |10\rangle\}$ and, with probability $q = 1 - p$, the unpolarized state of the subspace spanned by $\{|00\rangle, |11\rangle\}$. One can tune p to increase the weight of the first subspace which - in the presence of memory - is protected against noise. On the other hand, the amount of input information, measured by the von Neumann entropy $S(\rho^{\mathcal{Q}_1 \mathcal{Q}_2})$, is maximal when $p = q$. We can purify system $\mathcal{Q}_1 \mathcal{Q}_2$ by means of a two-qubit reference system $\mathcal{R}_1 \mathcal{R}_2$. We choose, for the system $\mathcal{R}_1 \mathcal{R}_2 \mathcal{Q}_1 \mathcal{Q}_2$, the following pure state:

$$|\psi^{\mathcal{R}_1 \mathcal{R}_2 \mathcal{Q}_1 \mathcal{Q}_2}\rangle = \sum_{i,j=0}^1 c_{ij} |ij^{\mathcal{R}_1 \mathcal{R}_2}\rangle |ij^{\mathcal{Q}_1 \mathcal{Q}_2}\rangle, \quad c_{ij} = \begin{cases} \sqrt{p/2} & \text{if } ij = 01, 10, \\ \sqrt{q/2} & \text{if } ij = 00, 11. \end{cases} \quad (28)$$

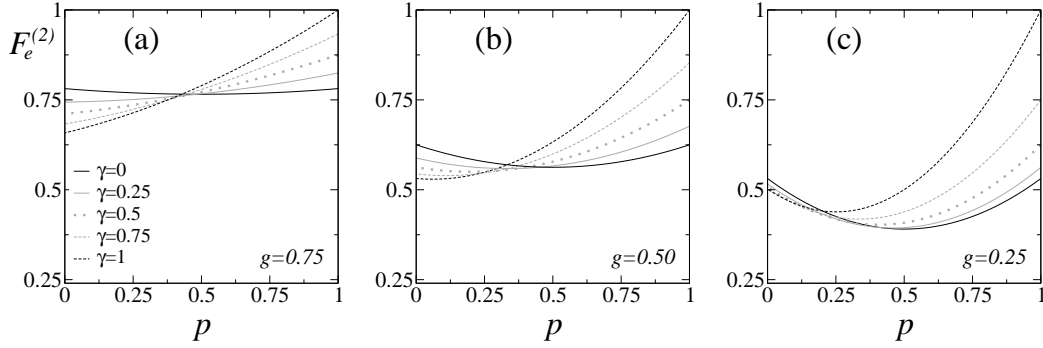


Fig. 2. Entanglement fidelity for two channel uses and input state (27), for different values of the dephasing factor g and the memory factor γ .

After the qubits \mathcal{Q}_1 and \mathcal{Q}_2 have crossed the channel, the system $\mathcal{R}_1\mathcal{R}_2\mathcal{Q}_1\mathcal{Q}_2$ is described by the density operator

$$\begin{aligned} \rho^{\mathcal{R}_1\mathcal{R}_2\mathcal{Q}'_1\mathcal{Q}'_2} &= (\mathcal{I}\mathcal{R}_1\mathcal{R}_2 \otimes \mathcal{E}_2^{\mathcal{Q}_1\mathcal{Q}_2}) (|\psi^{\mathcal{R}_1\mathcal{R}_2\mathcal{Q}_1\mathcal{Q}_2}\rangle\langle\psi^{\mathcal{R}_1\mathcal{R}_2\mathcal{Q}_1\mathcal{Q}_2}|) \\ &= \frac{1}{4} \begin{pmatrix} q & \dots & g\sqrt{pq} & \dots & g\sqrt{pq} & \dots & qh^+ \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g\sqrt{pq} & \dots & p & \dots & ph^- & \dots & g\sqrt{pq} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g\sqrt{pq} & \dots & ph^- & \dots & p & \dots & g\sqrt{pq} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ qh^+ & \dots & g\sqrt{pq} & \dots & g\sqrt{pq} & \dots & q \end{pmatrix}, \end{aligned} \quad (29)$$

where the dots stand for zeros (the matrix has dimension $2^4 \times 2^4$).

The entanglement fidelity is

$$F_e^{(2)} \equiv F_e(\rho^{\mathcal{Q}_1\mathcal{Q}_2}, \mathcal{E}_2) = \langle\psi^{\mathcal{R}_1\mathcal{R}_2\mathcal{Q}_1\mathcal{Q}_2}|\rho^{\mathcal{R}_1\mathcal{R}_2\mathcal{Q}'_1\mathcal{Q}'_2}|\psi^{\mathcal{R}_1\mathcal{R}_2\mathcal{Q}_1\mathcal{Q}_2}\rangle \quad (30)$$

and after some calculations we obtain

$$F_e^{(2)} = \frac{1}{2} [q^2(1+h^+) + p^2(1+h^-) + 4gpq]. \quad (31)$$

In Fig 2, we show several plots of the two-qubit entanglement fidelity (31). We can say that memory effects always improve this quantity, provided that the factor p is properly chosen. This effect is more evident for strong dephasing (see Fig. 2(c)).

We now turn our attention to the entropy exchange

$$S_e^{(2)} \equiv S_e(\rho^{\mathcal{Q}_1\mathcal{Q}_2}, \mathcal{E}_2) = S(\rho^{\mathcal{R}_1\mathcal{R}_2\mathcal{Q}'_1\mathcal{Q}'_2}) = - \sum_m \lambda_m \log_2 \lambda_m, \quad (32)$$

where the non-trivially equal to zero eigenvalues λ_m of the output density operator (29) are given by

$$\lambda_1 = \frac{1}{2}q(1-h^+), \quad \lambda_2 = \frac{1}{2}p(1-h^-) \quad \text{and} \quad \lambda_{3,4} = \frac{1}{4}(1+h^+ + h^- \pm \Delta_{34}), \quad (33)$$

with

$$\Delta_{34} = \sqrt{(1+qh^+ + ph^-)^2 + 4pq[4g^2 - (1+h^+)(1+h^-)]}. \quad (34)$$

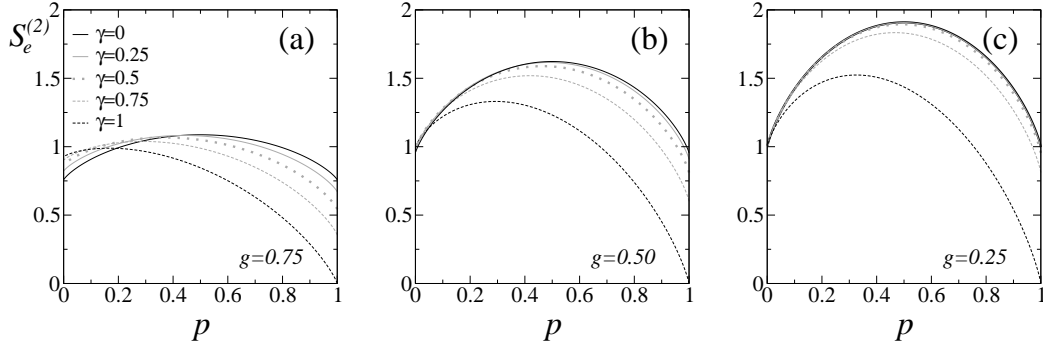


Fig. 3. Same as in Fig. 2 but for the entropy exchange.

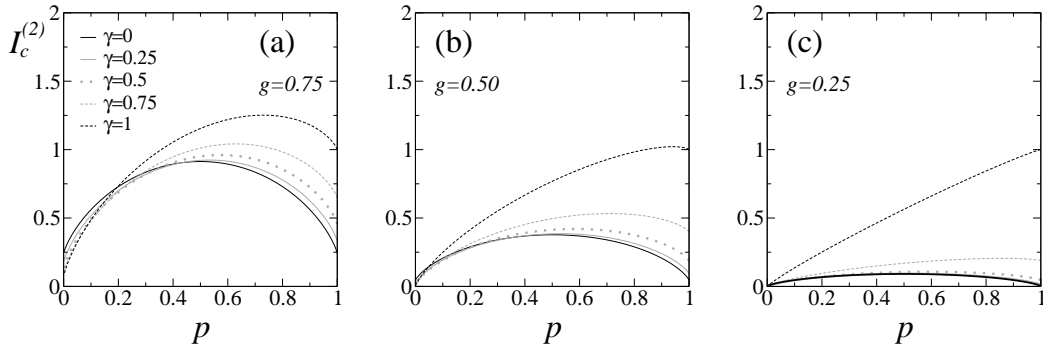


Fig. 4. Same as in Fig. 2 but for the coherent information.

As shown in Fig. 3, memory effects lower the entropy exchange, that is, memory reduces to some extent the information on the system acquired by the channel-environment. These plots are complementary to those in Fig. 2: as expected from the quantum Fano inequality [12], the information exchanged with the environment is small when the disturbance of the state is small, namely S_e is close to zero when F_e is close to one. The numerical data of Figs. 2-3 also show that S_e is close to one when F_e is close to zero.

Finally we turn to the coherent information

$$I_c^{(2)} \equiv I_c(\rho^{\mathcal{Q}_1 \mathcal{Q}_2}, \mathcal{E}_2) = S(\rho^{\mathcal{Q}'_1 \mathcal{Q}'_2}) - S_e^{(2)}. \quad (35)$$

Since the coherence terms in $\rho^{\mathcal{Q}_1 \mathcal{Q}_2}$ are equal to zero, the dephasing channel does not change this state and therefore $S(\rho^{\mathcal{Q}'_1 \mathcal{Q}'_2}) = S(\rho^{\mathcal{Q}_1 \mathcal{Q}_2}) = -p \log_2 \frac{p}{2} - q \log_2 \frac{q}{2}$ and

$$I_c^{(2)} = -p \log_2 \frac{p}{2} - q \log_2 \frac{q}{2} - S_e^{(2)}. \quad (36)$$

We can see from fig. 4 that coherent information is highly sensitive to memory effects. This sensitivity depends on the dephasing factor g . When dephasing is strong (g close to zero), strong memory effects are required to enhance the coherent information (see Fig. 4(c)), while for weaker dephasing (panels (a) and (b) in the same figure) it is possible to tune p to obtain noticeable enhancements also with weak memory effects. It is interesting to examine the limiting case of perfect memory ($\gamma = 1$) and to find, as a function of g , the value p_{opt} that maximizes the coherent information achievable for input state as in (27). Fig 5 shows that also in the regime of strong dephasing ($g < 0.5$) reliable transmission can be obtained, provided we exploit the decoherence-protected subspace $\{|01\rangle, |10\rangle\}$, that is, p_{opt} is close to one.

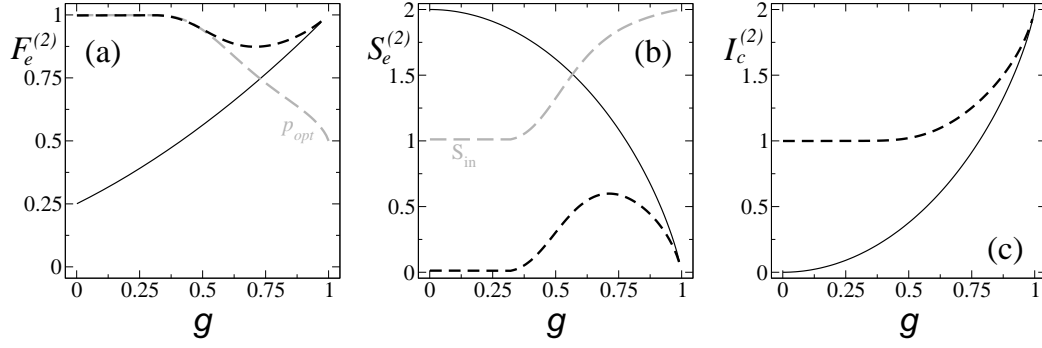


Fig. 5. Plots of (a) entanglement fidelity, (b) entropy exchange and (c) coherent information as functions of g for the input state (27), where p is chosen to maximize the coherent information (black dashed curves); we plot the corresponding curves for the memoryless case (black full curves). We also show in panel (a) p_{opt} as a function of g and in panel (b) the corresponding value of the input state entropy S_{in} (dotted grey curves).

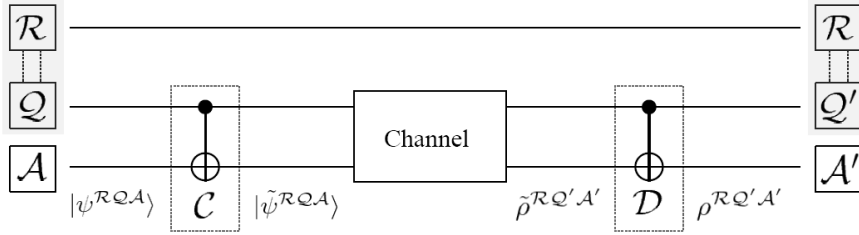


Fig. 6. Sketch of a coding-decoding scheme taking advantage of the correlation between two channel uses.

5.4 Coding-decoding scheme taking advantage of channel memory

We show that memory effects can be used to preserve entanglement in quantum information transmission. In particular, we consider an entanglement sharing protocol: Alice wish to send one qubit of a Bell pair (qubits \mathcal{R} and \mathcal{Q}) to Bob. The quantum channel (6),(11) randomizes the phase between the sent qubit (\mathcal{Q}) and the reference one (\mathcal{R}). In order to take advantage of memory effects, we follow the strategy sketched in Fig 6. We encode the sent qubit in a two-qubit system whose state resides in the subspace (spanned by $\{|01\rangle, |10\rangle\}$) resilient to errors in the presence of memory effects.

Without loss of generality we assume that the initial Bell state of the entangled pair is $|\psi^{\mathcal{RQ}}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\phi^+\rangle$. The coding-decoding protocol is performed in the following way:

(a) We prepare an ancillary qubit \mathcal{A} in the state $|1\rangle^{\mathcal{A}}$, such that the whole system \mathcal{RQA} is initially in the state

$$|\psi^{\mathcal{RQA}}\rangle = |\psi^{\mathcal{RQ}}\rangle \otimes |1\rangle^{\mathcal{A}} = \frac{1}{2}(|001\rangle + |111\rangle). \quad (37)$$

(b) The encoding operation \mathcal{C} is a controlled-not gate acting on the system \mathcal{QA} , where \mathcal{Q} is the control qubit:

$$|\tilde{\psi}^{\mathcal{RQA}}\rangle = (\mathbf{1}^{\mathcal{R}} \otimes \text{CNOT}^{\mathcal{QA}})(|\psi^{\mathcal{RQA}}\rangle) = \frac{1}{2}(|001\rangle + |110\rangle). \quad (38)$$

As a result, we encode the system \mathcal{RQA} into a GHZ state, in such a way that the subsystem \mathcal{QA} resides in the subspace spanned by $\{|01\rangle, |10\rangle\}$. This is the case when the entangled state

$|\psi^{\mathcal{RQ}}\rangle$ is any state of the Bell basis:

$$\begin{aligned} |\phi^\pm\rangle &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) \longrightarrow \frac{1}{\sqrt{2}}(|001\rangle \pm |110\rangle), \\ |\psi^\pm\rangle &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) \longrightarrow \frac{1}{\sqrt{2}}(|010\rangle \pm |101\rangle). \end{aligned} \quad (39)$$

(c) We send qubits \mathcal{Q} and \mathcal{A} through the channel. We call $\tilde{\rho}^{\mathcal{RQ}'\mathcal{A}'}$ the density operator describing the state that arises from the channel transmission:

$$\tilde{\rho}^{\mathcal{RQ}'\mathcal{A}'} = (\mathcal{I}^{\mathcal{R}} \otimes \mathcal{E}^{\mathcal{QA}})(|\tilde{\psi}^{\mathcal{RQA}}\rangle\langle\tilde{\psi}^{\mathcal{RQA}}|). \quad (40)$$

(d) The decoding operation \mathcal{D} extracts the state of system \mathcal{RQ} from the one of \mathcal{RQA} . To this aim we apply another controlled-not gate to the system \mathcal{QA} , where \mathcal{Q} is - as above - the control qubit. This operation disentangles systems \mathcal{RQ} and \mathcal{A} :

$$\rho^{\mathcal{RQ}'\mathcal{A}'} = (\mathbf{1}^{\mathcal{R}} \otimes \text{CNOT}^{\mathcal{QA}}) \tilde{\rho}^{\mathcal{RQ}'\mathcal{A}'} (\mathbf{1}^{\mathcal{R}} \otimes \text{CNOT}^{\mathcal{QA}}) = \rho^{\mathcal{RQ}'} \otimes |1\rangle^{\mathcal{A}}, \quad (41)$$

where

$$\rho^{\mathcal{RQ}'} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) + \frac{h^-}{2}(|00\rangle\langle 11| + |11\rangle\langle 00|). \quad (42)$$

The fidelity of the final state $\rho^{\mathcal{RQ}'}$ is

$$F = \langle \phi^{\mathcal{RQ}} | \rho^{\mathcal{RQ}'} | \phi^{\mathcal{RQ}} \rangle = \frac{1 + h^-}{2}. \quad (43)$$

This is also the entanglement fidelity F_e^c when the initial state of \mathcal{Q} is $\rho^{\mathcal{Q}} = \frac{1}{2}\mathbf{1}$ and when the above coding-encoding scheme is used. We compare this value with the entanglement fidelity (17), obtained when the qubit \mathcal{Q} is simply sent down the channel. Therefore, the above coding-encoding strategy is useful when memory effects are strong enough, namely when

$$F_e^c \geq F_e \quad \Rightarrow \quad h^- = g^{2(1-\gamma)} \geq g \quad \Rightarrow \quad \gamma \geq 0.5. \quad (44)$$

In spite of its simplicity, coding/decoding schemes similar to the one described in this section can be useful to protect information in the presence of memory effects [27].

6 Conclusions

In this paper we have shown that, already for two channel uses, memory effects can deeply modify the behaviour of a quantum Hamiltonian dephasing channel with respect to the memoryless limit. It is relevant that memory, already with two channel uses, can be used to enhance the channel capability to transmit coherent quantum information. This result may be of interest for present-day few-qubit experimental implementations of quantum hardware.

We thank A. Mastellone and E. Paladino for discussions. G.B. acknowledges support from MIUR-PRIN 2005 (2005025204). A.D. and G.F. acknowledge support from the EU-EuroSQIP (IST-3-015708-IP) and MIUR-PRIN2005 (2005022977).

References

1. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge 2000)

2. G. Benenti, G. Casati and G. Strini, *Principles of Quantum Computation and Information*, Vol. II: Basic tools and special topics (World Scientific, Singapore 2007)
3. C. H. Bennet and P. Shor, IEEE Transactions on Information Theory **44** (1998) 2724
4. K. Banaszek, A. Dragan, W. Wasilewski, and Radzewicz, Phys. Rev. Lett **92** (2004) 257901
5. E. Paladino, L. Faoro, G. Falci, and R. Fazio, Phys. Rev. Lett. **89** (2002) 228304; E. Paladino, L. Faoro, and G. Falci, Adv. Solid State Phys. **43** (2003) 747; E. Paladino, A. Mastellone, A. D'Arrigo, and G. Falci, *Realizing Controllable Quantum States* (Takanayanagi H and Nitta J, World Scientific, London 2005) 282
6. C. Macchiavello and G. M. Palma, Phys. Rev. A **65** (2002) 050301
7. A. D'Arrigo, G. Benenti, and G. Falci, New J. Phys. **9** (2007) 310
8. D. Leung, L. Vandersypen, X. Zhou, M. Sherwood, C. Yannoni, M. Kubinec, and I. Chuang, Phys. Rev. A **60** (1999) 1924
9. Y. Makhlin and A. Shnirman, Phys. Rev. Lett. **92** (2004) 178301; G. Falci, A. D'Arrigo, A. Mastellone, and E. Paladino, Phys. Rev. Lett. **94** (2005) 167002; G. Ithier, E. Collin, P. Joyez, P.J. Meeson, D. Vion, D. Esteve, F. Chiarello, A. Shnirman, Y. Makhlin, J. Schrieffer, and G. Schön, Phys. Rev. B **72** (2005) 134519
10. B. Schumacher, Phys. Rev. A **51**, (1995) 2738
11. H. Barnum, M. A. Nielsen, and B. Schumacher, Phys. Rev. A **57**, (1998) 4153
12. B. Schumacher, Phys. Rev. A **54**, (1996) 2614
13. H. Barnum, E. Knill, and M. A. Nielsen, Information Theory, IEEE Transactions, **46** (2000) 1317
14. B. Schumacher and M. A. Nielsen, Phys. Rev. A **54** (1996) 2629
15. C. E. Shannon, The Bell System Technical Journal **27** (1948) 379 and 623.
16. T. M. Cover and J. A. Thomas, *Elements of Information Theory* (Wiley, New York 1991).
17. S. Lloyd, Phys. Rev. A **55** (1997) 1613
18. D. Kretschmann and R. F. Werner, Phys. Rev. A **72** (2005) 062323
19. H. Barnum, M. A. Nielsen, and B. Schumacher Phys. Rev. A **57** (1998) 4153
20. P. W. Shor 2002, *The quantum channel capacity and coherent information*, lecture notes, MSRI Workshop on Quantum Computation (2002)
21. I. Devetak, IEEE Trans. Inf. Theory **51** (2005) 44
22. P. Hayden, M. Horodecki, J. Yard, and A. Winter 2007, quant-ph/0702005
23. U. Weiss, *Quantum dissipative systems* (2nd Ed.) (World Scientific, Singapore 1999)
24. I. Devetak and P. W. Shor, quant-ph/0311131
25. V. Giovannetti, J. Phys. A: Math. Gen. **38** (2005) 10989.
26. M. M. Wolf and D. Pérez-García, Phys. Rev. A **75** (2007) 012303
27. T. Yamamoto, R. Nagase, J. Shimamura, S. K. Ozdemir, M. Koashi, and N. Imoto, New J. Phys. **9** (2007) 191